Forbearance and Broken Credit Cycles

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Abstract

This paper studies how post-financial-crisis economy differs from its pre-crisis period, by providing a microeconomic model of asset price, aggregate output and banks' strategic behaviour. Banks in this model avoid liquidating non-performing loans to make up their balance sheets (i.e. forbearance or zombie lending), when realising loan losses would reduce their capitals lower than the regulatory threshold and force the banks to go bankrupt. But the forbearance comes at a macro-economic cost. Knowing that banks have to liquidate forborne non-performing loans some day, they expect the decrease of collateral asset value in the future and reduce new lending to productive firms. Forbearance thus decelerates de-leverage of less-productive sectors while it accelerates de-leveraging productive sectors, leading to the reduction of aggregate output. However, we also identify the possibility of output-boosting forbearance when the economy has a strong 'financial accelerator effect'. By studying both banks' incentive of forbearance and its impacts on macro-economy and banking systems, this paper provides a rich model explaining some of post-crisis phenomena, such as low productivity, slow de-leverage in specific sectors, sluggish new lending, excessively resilient property prices, and broken correlation between asset price and output growth. The paper concludes with a welfare analysis of forbearance and a discussion on optimal policy measures.

Keywords: Forbearance, Credit cycles, Productivity, Strategic banks
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1 Introduction

This paper studies how post-financial-crisis economy differs from its pre-crisis period, by providing a microeconomic model of asset price, aggregate output and banks' strategic behaviour. Banks in this model avoid liquidating non-performing loans to make up their balance sheets (i.e. forbearance or zombie lending), when realising loan losses would reduce their capitals lower than the regulatory threshold and force banks to go bankrupt. This is a rational behaviour for the banks, but the forbearance comes at a macro-economic cost. Since banks know that they have to liquidate forborne non-performing loans some day, they expect the decrease of collateral asset value in the future. They therefore raise the haircut of collateral asset when they make new loans, which reduces new lending to productive firms. The forbearance thus decelerates de-leverage of non-performing sectors while it accelerates the de-leveraging of productive sectors, leading to the reduction of aggregate output, and the reduction of banks' profitability as well. By studying both banks' incentive of forbearance and its impacts on macro-economy and banking systems, this paper can provide an explanation some of post-crisis phenomena, such as low productivity, slow de-leverage in specific sectors, sluggish new lending, excessively resilient property prices, and broken correlation between asset price and output growth. The paper concludes with a welfare analysis of forbearance and a discussion on optimal policy measures.

Although financial markets are recovering from the crisis in 2008, it does not necessarily mean that the real economies go back to the status in pre-crisis period. We still have several persistent aftereffects of the crisis. Productivity was lowered after the crisis (Hayashi and Prescott, 2000, show the TFP decline in Japan after the market crash, and Hughes and Saleheen, 2012, show a similar phenomenon globally). De-leverage of banking sector was slower (Kobayashi et. al., 2003, show lending to non-performing sector was rather increasing after the market crash. Loan amount of real estate sector and others have also not decreased compared with manufacturing sectors in the UK). Land price (or property price) showed limited adjustment after the rapid boom in pre-crisis period (UK house price increased by 140% from 2000 till 2007 while decreased only by 10% from 2007 till 2012. Japanese land price marked its lowest growth -8.5% in 2005 although the market crashed in 1991). In addition, the positive correlation between output and land price, predicted by Kiyotaki's and Moore's (1997) seminal paper, was broken down after the crisis in Japan (the correlation coefficient from 1960 till 1991 was 0.51, while the coefficient turns to be negative -0.15 from 1991 till 2005 when Japanese banks finished resolving their non-performing loan problem). See Figure 1. (The land price is real.) It should be also noted that these phenomena are not observed in the US after the recent crisis.

The aim of the paper is explaining those phenomena as a result of banking crisis, especially forbearance. There are many papers studying why financial crisis occurs, or why credit booms occurred and were followed by the bust (e.g. Jeanne and Korinek, 2010, Lorenzoni, 2008,
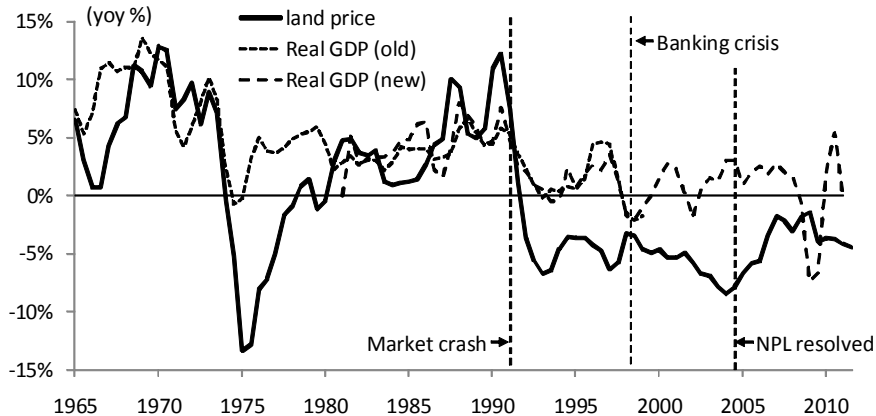
Figure 1: GDP growth and land price growth (Japan)

Geanakoplos, 2010, and many others). But the study on post-crisis phenomena is still scarce, to the author's knowledge (exceptions are Caballero et. al., 2008, and Philippon and Schnabl, 2013). This paper tries to shed light on another aspect of post-crisis economy, to see why the phenomena cited above occurs after banking crisis.

This model particularly focuses on forbearance as a mechanism in post-crisis economy. Forbearance, sometimes called zombie lending (Caballero et. al., 2008), or evergreening loans (Peek and Rosengren, 2005), is the phenomenon that banks do not liquidate non-performing loans (or collateral assets seized from the borrowers) and leave those on their balance sheets. Banks choose forbearance because it is too costly to realise loan losses, which would reduce their capitals lower than the regulatory threshold and trigger their bankruptcies. By postponing the liquidation of non-performing borrowers (and their collateral assets), bank can contain the plunge of asset price and the following significant deterioration of balance sheets, at least for the time being.

The forbearance affects the macroeconomy and the banking system in the followingn paths. Postponed liquidation contains the plunge of asset price which could have been significant when the economy experiences a negative macro shock. This helps banks hide their true loan losses for the time being as they do not have to realise the insufficient collateral values to cover the loans. But all the players in the economy understand that banks cannot postpone the liqudation forever, and form an expectation that the collateral value will drop in the future. As banks (lenders) see the expected value of collateral at the maturity date of a loan they extend, the expectation tightens the credit constraint of the productive firms with funding needs. In other words, this model endogenises leverage ratio: while the previous literature is prone to assume that the ratio is fixed and borrowers' net wealth thoroughly determines the loan size, this model

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shows that lending amount could decrease when borrowers' net wealth is increased when banks anticipate the decline of collateral value. This is a core feature of the model explaining the 'broken credit cycles' mentioned above. The tightened credit constraint reduces the output of the economy since productive firms invest less, and lowers the profitability of whole banking sector as new profitable lending opportunities decrease.

As we study the interaction between asset price, output and banks' behaviour, a natural starting point of the discussion would be Financial Accelerator models. This model is a reduced-form version of Kiyotaki and Moore (1997, KM hereafter). The most relevant paper is Krishnamurthy (2003) introducing credit-constrained insurers to two-periods reduced-form credit cycles model. Jeanne and Korinek (2010) and Lorenzoni (2008) also build similar models. There are a couple of critical differences distinguishing the paper from these reduced-form financial accelerator models. First one is studying price dynamics. For example in Krishnamurthy's model asset price q_t is determined with Financial Accelerator effect only once at t=1 (the other two models above have a similar feature). No amplification mechanism works at t=0 and 2. This is not a convenient feature to consider price dynamism: as described above, it is a key feature of the model that anticipated asset price decline reduce the leverage of the economy. Therefore asset price should be determined under the identical setup at least twice. As we see below, this is not a trivial extension. Second, this model explicitly introduces the defaults of borrowing firms and banks. Banks' loss given default of the firms is endogenously determined.

Another relevant literature on forbearance is Kocherlakota and Shim (2007). Kocherlakota and Shim (2007) study the optimality of regulatory forbearance, i.e. forbearance by the social planner. In their model, the probability distribution of collateral value plays a crucial role. Lenders are tempted to liquidate their borrowers when collateral value is low since low collateral value cannot incentivise the borrowers without additional reward. The lenders would not liquidate them (forbearance) if the government rescures the lenders by compensating the loss (the reward), but the government is not willing to commit this if the probability of low collateral value is high. The model discussed here is different from their work in various ways. Collateral value is endogenously determined, so as the cost of liquidation. Forbearance is the choice of banks, not the government. Forbearance is not committed ex-ante, although correctly anticipated by strategic players. These features provide rich implications on forbearance as we see below.

Theoretically, this paper can be also seen as a model in which banks internalise the externalities of liquidating zombie borrowers (to some extent), in contrast to Jeanne and Korinek (2010) or Lorenzoni (2008) in which externality plays an important role. Jeanne and Korinek (2010), for instance, argue borrowers create excessive leverage since they are not aware of (positive) externality of holding liquid net worth. In this model, on the other hand, banks are well aware of the externality of liquidating bad borrowers at once and coordinate with other banks to do

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forbearance. This reduces the depth of a bust in the asset market, but worsens the slump of the output. In addition, anticipating the possibility of forbearance which reduces the pain under recession, banks loosen their credit conditions ex-ante, which boosts the pre-crisis economy and deepens the loss of output in the post-crisis economy, i.e. forbearance could amplifies boom-bust cycles.

The paper is structured as follows. Section 2 introduces the benchmark model, which banks do not do forbearance. Section 3 introduces forbearance into the model and section 4 discusses the social welfare and possible policy options. Section 6 concludes, with the discussion considering the reasons why US did not experience forbearance and the post-crisis phenomena cited above. Appendix provides the detail of mathematical proofs.

2 Benchmark Model

In this section, we will see the baseline model without forbearance. We first clarify the assumptions and timeline. We then derive firms' demand function of land, which is constrained by collateral constraint imposed by banks, as well as dealers' demand function. We find a unique, or multiple, competitive equilibrium, and see how the equilibrium price (and equilibrium firms' investment) behaves throughout the period. In this baseline model, equilibrium price and output always have positive correlation. The equilibrium price q_t follows $I(1)$ process with a small negative drift $E_t[q_{t+1}] \simeq q_t$ since the firms' expected wealth at t+1 is equal to the current wealth at t. The negative drift emerges from the non-linearity of the model and not essential.

2.1 Assumptions

There are three sets of players in the economy: firms, dealers and banks. They are all risk neutral, and are born at t=0 and die at t=3. The first three periods represent pre-crisis (t=0), during crisis (t=1), and post-crisis (t=2). At t=3 all players consume everything they have, and die. Firms and dealers are continuum players with measure one. The number of banks is finite $N \in \mathbb{R}_{++}$ but would be large. For the notational simplicity, we describe equations as if $N \to +\infty$. The economy has only one asset, denoted as land. The supply of land is fixed at K. Land is initially supplied by an outsider of the economy at the beginning of t=0, and the same outsider purchases at a price (see below) at t=3 when all players die. Some players can produce a consumption good (wealth) using land, and also they can pledge land as collateral to borrow wealth from banks.

At each period, firms and dealers purchase land to initiate their production opportunities. Firms are assumed to be credit constrained: i.e. they do now have sufficient endowment to purchase optimal amount of land and banks provide collateralised loans to the firms (in exchange of land). Since banks have no production opportunity, banks' assets are thoroughly loans to the firms. The dealers are assumed to have an access to an unconstrained funding source, as well as the banks.

Next period, firms obtain harvests from the production opportunities. A fraction of firms fail to harvest any (and go bankrupt - this is an idiosyncratic shock). The rest of the firms obtain harvest and the amount can take two values, high or low (macro shock). The dealers' production is non-stochastic, but the expected productivity of the firms is higher than that of the dealers. Banks cannot obtain any repayment from the failed firms and seize collateralised land instead. Surviving firms, the dealers and the banks seizing collateral land sell all their land holding, and the firms and the banks repay all their debts. At the same time, the surviving firms and the dealers purchase land for the new production opportunities. The Walrasian Auctioneer sets the land price to clear the market.

This process is repeated three times (t=0,1 and 2) and at t=3, the final period, all the players sell their land holding to the outsider of the economy at the equilibrium price at t=2, and die after they consume all the wealth.[1] This is an adhoc assumption, but we will see below that this does not necessarily lose generality much in fact.

We will find equilibria specifying land price q_t, firms' land holding k_t^f, dealers' land holding k_t^b, banks' lending amount D_t and required repayment R_t for $t = \{0, 1, 2\}$.

2.2 Firms' problem

The economy has continuum firms with measure one. They are ex-ante identical, risk neutral players possessing a profitable investment opportunity. The opportunity requires a production capital, denoted as land hereafter (land does not depreciate). By purchasing land at a period t, the firms can produce a_{t+1} per unit of production capital with probability $1-\gamma$. With probability γ firms fails to harvest any. Surviving firms' productivity a_{t+1} is a stochastic variable taking $a_{t+1} \in \{a_H, a_L\}$ with probability π and $1 - \pi$ respectively, updated at each period (i.i.d. over time). The choice of a_{t+1} is a macro shock, i.e. it is the same across firms. a_{t+1} is realised at t+1, but firms (and banks) have expectation of a_{t+1} at t: the expectation is denoted as $E_t[a_{t+1}]$.

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[1]Otherwise land price at t=3 should be $q_3 = 0$ since the asset becomes useless for the players who no longer produce (see Krishnamurthy (2003)). Since expected price growth plays an important role, $q_3 = 0$ is inconvenient in this model.

Note that the firms are exposed to two shocks, idiosyncratic one (with probability γ) and macro one (with probability π).

All the firms receive an initial endowment ω_0 at the beginning of t=0. Firms purchase land k_0^f, and lever their position by borrowing from banks pledging purchased land as collateral. It is assumed that ω_0 is too small so that firms cannot start investment projects without borrowing money from banks. I assume $E_t[a_{t+1}]$ is sufficiently high so that the leverage continues until the following budget and collateral constraints bind (note that the firms' production function is linear).

We assume that firms can take two "moral hazard" actions to justify collateralisation of loans and the haircut of collateral. First one is walking away with borrowed wealth. Therefore banks have to provide collateralised loans only. Second, firms can take a cheating investment project, which provides a private benefit B to the firms but the default probability is fixed at $\gamma = 1$. Banks thus have to charge haircut over collateral asset, in order to let the firms invest their own wealth to the investment project that is larger than B. In the following argument we treat h as an exogenous parameter since h is anyway the function of exogenous parameter B. Practically, h would represent a loan-to-value ratio (LTV) cap applied by public authorities. Note that firms cannot walk way after they harvest. This is because the harvest is directly stored at the lending banks' current account.[2]

Budget constraint of the firms is

$$q_t k_t^f \leq D_t + \omega_t \tag{1}$$

i.e. the purchase of land is capped by the loan size plus endowment. And the loan size is capped by the following collateral constraint

$$D_t \leq (1-h)E_t\left[q_{t+1}\right]k_t^f \tag{2}$$

Note that collateral does not cover firms' interest payment since we have assumed that firms can pledge the repayment above.

Normally, haircut is applied to the current asset value, not the expected asset value. To incorporate the reality, we define "actual haircut" $\tilde{h}$ so that:

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[2]We need this assumption since we assume very high a_t and that banks take all excess surplus. If collateral has to cover future (possible) repayment, firms' credit constraint is tightened as firms become profitable. This is less intuitive and less convenient for our modelling here. And in reality, LTVs are calculated based on principals, not on the total repayment value.

$$(1-h)E_t\left[q_{t+1}\right] = (1-\tilde{h})q_t$$
$$\tilde{h} = 1-(1-h)\frac{E_t\left[q_{t+1}\right]}{q_t} \tag{3}$$

The equation (2) can be written as:

$$D_t \leq (1-\tilde{h})q_t k_t^f$$

and if the budget and collateral constraints are binding,

$$D_t = q_t k_t^f - \omega_t = (1-\tilde{h})q_t k_t^f$$

since $q_t k_t^f = \frac{\omega_t}{\tilde{h}}$,

$$D_t = q_t k_t^f - \omega_t = \frac{1-\tilde{h}}{\tilde{h}}\omega_t \tag{4}$$

As long as $\tilde{h}$ is independent from ω_t, lending amount is a linearly increasing function of firms' wealth ω_t (we will see below). The firms' wealth is updated along with the following transition function.

$$E_t[\omega_{t+1}] = \pi\left(1-\gamma\right)\left(\omega_t + D_t - q_t k_t^f + a_{t+1}^H k_t^f + q_{t+1}^H k_t^f - R_t\right)$$
$$+ (1-\pi)\left(1-\gamma\right)\left(\omega_t + D_t - q_t k_t^f + a_{t+1}^L k_t^f + q_{t+1}^L k_t^f - R_t\right) \tag{5}$$

where R_t is the repayment amount (for the time being we assume that this is not state-contingent). With probability γ firms cannot harvest any, but still they have their production capital. Firms will sell land to repay the loan, or land would be seized by banks when firms fail to harvest. Note also that the first three terms in the brackets are zero since the credit constraints are binding.

2.3 dealers' problem

We define another set of players dealers: they have unconstrained access to a funding source outside of the economy at the interest rate r, and can purchace land k_t^b for their investment opportunity producing $k_t^b\left(A - k_t^b\right)$ at the next period. A is constant and $A < E[a]$, i.e. dealers are less productive than firms, and $A = 2K$ to ensure that $k_t^b\left(A - k_t^b\right)$ is monotonically increasing everywhere.

We further assume that dealers cannot observe firms' behaviour and their objective functions, so they create their ex-ante belief on the next period land price q_{t+1} randomly around the current price q_t. I.e. they are assumed to be noise traders as assumed by e.g. Kyle (1985) and on average they expect $Et[q_{t+1}] = q_t$.

The dealers' payoff function Π, to be maximised, is:

$$\Pi_t = f\left(A, k_t^b\right) - rq_t k_t^b$$

The FOD of $E\left[\Pi_{t+1}\right]$ wrt k_t^b is

$$\frac{\partial E\left[\Pi_{t+1}\right]}{\partial k_t^b} = A - 2k_t^b - rq_t = 0$$

And the banks' demand function is

$$k_t^b = \frac{A}{2} - \frac{1}{2}rq_t$$

For the notational convenience, we normally define the dealers' demand function on the domain of k_t^f: therefore the dealers' demand function is horizontally flipped and described as an upward sloping curve.

2.4   Bank's problem

Banks are the only players who can write loan contracts to firms. Banks can obtain funding for the lending from the outside funding source at the interest rate r. Each bank lends to many firms, but no bank is allowed to diversify their loan portfolio.[3] Each bank's default probability of the borrowers is perfectly correlated, so that with probability γ banks fail to receive repayment at all, and will be repaid in full with probability $1 - \gamma$. This assumption is clearly unrealistic but this is the simplest way to introduce banks' default (explained below), as assumed by Holmstrom and Tirole (1997). Each firms can only borrow from a bank. The matchings between banks and firms are reshuffled at each period.

Banks write loan contracts specifying the loan size D_t, and repayment amount R_t: i.e. they have perfect negotiation power over borrowing firms, by making a take-it-or-leave-it offer to firms. They can thus take all excess surplus of firms' investment project.

If banks fail to be repaid, they seize collateral assets and liquidate those at the asset market. If the liquidation value (determined endogenously in the asset market) is lower than the lending

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[3]I.e. each bank has monopolistic power against borrowers. This can be justified, e.g. by costly monitoring.

amount, then those banks incur credit losses. If the loss is large enough for their capital level W_t to go below a threshold $\bar{W}$, public authority forces the bank to close their business, which is very costly for the bank managers (incurring a non-pecuniary cost X). The failed banks are eliminated from the economy, and surviving banks replace the loans that the failed banks extended to the borrowers immediately. I.e. there is no real economic cost of bank failure in this model (depositors lending to the banks would incur a loss, but we do not consider the problem).

The initial capital W_0 is endowed at the beginning of $t = 0$. Note that a bank's capital W_t does not restrict banks' lending activity: i.e. there is no capital adequacy ratio constraint in the economy. The threshold $\bar{W}$ is introduced only to introduce the bank failure. Note also that the banks' failure do not restrict the aggregate loan size D_t by the same reason. Loan amount is restricted by firms' credit constraint only — i.e. the real economic cost of bank failure is zero at this moment.[4]

Banks choose $\{D_t, R_t, r_t\}$ maximising their $E_t[W_3]$ at each period t. This is equivalent to maximise $E_t[W_{t+1}]$, because it is optimal in the long run for the banks to take all excess surplus at each period. It would not be impossible that banks may find it optimal to leave some excess surplus on the hand of firms at t=0 and t=1, since irms with larger net wealth can borrow more in the following period(s), which increases profitable lending opportunity for the banks. But the possibility has been eliminated since borrowers are reshuffled at each period. A bank has no incentive to leave some profit for the borrowers if all the other banks take all excess surplus from their borrowers (as the probability to be matched with the same borrower is very small), and the bank has no incentive to leave some profit even if all the other banks leave some profits for their borrowers since the bank can enjoy free lunch. Therefore being myopic is the unique Nash equilibrium.

The banks' capital transition function, given that the banks survive at the period t (i.e. $W_t \geq \bar{W}$) is defined as follows:

$$E_t[W_{t+1}] = W_t + E_t\left[(1-\gamma)R_t + \gamma q_{t+1}k_t^f - (1+r_t)D_t + (1-I_{t+1})X\right]$$

The first term in the blacket on the RHS is the repayment revenue, the second term is the liquidation value of failed borrowers' collateral, and the third term is the repayment amount to the dealers. The last term is the expected cost of the bank's failure by $W_{t+1} < \bar{W}$. I_{t+1} is the index variable: $I_{t+1} = 1$ when $W_{t+1} \geq \bar{W}$ and $I_{t+1} = 0$ otherwise. X is the bank managers' private cost of bank closure.

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[4] This part could be revised later to introduce the cost of bank failure.

The banks maximise this transition equation subject to the firms' and banks' participation constraints:

$$E_t[\omega_{t+1}] \geq \omega_t$$
$$E_t[W_{t+1}] \geq W_t$$

The assumption of the sufficiently high firms' profitability $E[a]$ ensures that the banks want to maximise D_t where firms' credit constraint binds. This assumption also ensures that the banks' participation constraint above is satisfied even if the expected cost of bank closure $E_t[(1 - I_{t+1})X]$ is large.

The banks with superior negotiation power choose the maximum possible R_t satisfying the firms' participation constraint. The LHS of the participation constraint (equation 5) is:

$$
\begin{aligned}
E_t[\omega_{t+1}] &= \pi(1-\gamma)\left(a_{t+1}^H k_t^f + q_{t+1}^H k_t^f - R_t\right) \\
&\quad + (1-\pi)(1-\gamma)\left(a_{t+1}^L k_t^f + q_{t+1}^L k_t^f - R_t\right)
\end{aligned}
$$

Following to the optimal debt contract, we assume that R_t is constant and the firms take all the stochastic residuals as retained earnings. Denote that the expected retained earning conditional on the firms' survival as T_t. Then we can rewrite the equation as follows:

$$
E_t[\omega_{t+1}] = (1-\gamma)\left[
\begin{array}{c}
T_t + \pi\left\{\left(a_{t+1}^H - E_t[a_{t+1}]\right)k_t^f + \left(q_{t+1}^H - E_t[q_{t+1}]\right)k_t^f\right\} \\
+ (1-\pi)\left\{\left(a_{t+1}^L - E_t[a_{t+1}]\right)k_t^f + \left(q_{t+1}^L - E_t[q_{t+1}]\right)k_t^f\right\}
\end{array}
\right]
$$

where $\left(a_{t+1}^H - E_t[a_{t+1}]\right)k_t^f$ is the (expected) income gain and $\left(q_{t+1}^H - E_t[q_{t+1}]\right)k_t^f$ is the (expected) capital gain. The equation assumes that the firms will use 100% of capital gain from their land holding to increase their investment. This is, however, not very realistic. Many non-financial firms record their assets in book value, not market value, to segregate their profits from market price volatility which is irrelevant to their own business. To incorporate this, we introduce one more parameter $\eta \in [0,1]$: the firms' net wealth is increased (or decreased) by the fraction η of the capital gain.

For the time being, we assume relatively small η, since this ensures the uniqueness of the equilibrium. In addition, since we mainly study post crisis phenomena, it is unlikely that firms actively mark-to-market their production capital during recession. We will come back to the case when η is large in Section 3.4.

For the notational convenience, we use the following transition function, given a_{t+1}, throughout the paper.

$$
\begin{aligned}
\omega_{t+1}|_{a_{t+1}} &= (1-\gamma)\left\{T_t + \left(a_{t+1}^H - E_t\left[a_{t+1}\right]\right)k_t^f + \eta\left(q_{t+1} - E_t\left[q_{t+1}\right]\right)k_t^f\right\} \\
&= (1-\gamma)\left\{\psi(T_t) + \eta q_{t+1}k_t^f\right\}
\end{aligned}
\tag{6}
$$

$\psi(T_t)$ is the constant component at t+1. Note that $\psi(T_t)$ can be negative for a large η. Note also that $E_t[\omega_{t+1}] = (1-\gamma)T_t$, irrespective of η.

2.5 Firms' demand function of land

Now we can define firms' demand function of land, given the banks' optimal choice of D_t and R_t. Firms try to maximise their final wealth ω_3, under the budget and credit constraints defined above. And the assumption of the firms' sufficiently high profitability ensures the corner solution, i.e. the firms' budget and credit constraints, the equations (1) and (2). By rewriting these equations, we have:

$$
q_t k_t^f \le (1-h)E_t\left[q_{t+1}\right]k_t^f + \omega_t
$$

And from the equations (6) and (3), we have the following firms' inverted constrained demand function as follows:

$$
k_t^f = \frac{\omega_t}{\tilde{h}q_t} = \frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k_{t-1}^f\right\}}{q_t - (1-h)E_t\left[q_{t+1}\right]}
$$

If the firms' demand is not constrained, the demand function becomes horizontal as the production opportunity is constant return to scale (we have assumed sufficiently large $E\left[a\right]$ to ensure that the uncontrained equilibrium does not emerge. The collateral constraint does not bind anywhere $q_t \le (1-h)E_t\left[q_{t+1}\right]$ (note that the actual haircut $\tilde{h}$ becomes negative in this region). Therefore the demand curve kinks twice, as Figure 2(a) shows. The curve cannot be lower than $q_t = (1-h)E_t\left[q_{t+1}\right]$.

It is important that the firms' net wealth ω_t is the positive function of the spot land price q_t, and therefore could boost the firms' demand k_t^f. This is exactly what financial accelerator models assume. Furthermore, η controls the strength of the financial accelerator in this model.

Note also that figure 2(a) assumes small η: if η is sufficiently high, the demand curve could be upward sloping (see section 3.4). Small η ensures a downward sloping demand curve since

the positive 'wealth effect', i.e. a higher asset price boosts the firms' net wealth, is outweighed by the 'price effect', i.e. a higher price reduces the firms' purchasing power. Only at t=0, endowed ω_0 is constant and the constrained demand function is decreasing and convex to the origin for sure. And at t=2 we assume $q_3 = q_2$ thus $\tilde{h} = h$. The shapes of those demand functions (at t=1 and t=2) are defined as follows:

Lemma 1 *At t=1, for a given and fixed $E_t[q_{t+1}]$, the demand function is downward sloping and convex to the origin if $E_{t-1}[q_t] < (1-h)E_t[q_{t+1}]$. If $E_{t-1}[q_t] > (1-h)E_t[q_{t+1}]$, there exists $\hat{\eta}$ such that for $^{\forall}\eta < \hat{\eta}$ the function is downward sloping and convex to the origin, and vice versa. At t=2 where $q_{t+1} = q_t$, for any $\eta < \check{\eta}$ the demand curve is downward sloping and convex to the origin, and otherwise upward sloping and convex.*

See the appendix for the proof. When η is small, the financial accelerator effect becomes weak and normal price effect (for a given budget, purchasing amount is negatively correlated to price) dominates the financial accelerator. The curve is convex to origin since firms are leveraged (the curve becomes linear if $h = 1$). But if η is high enough, the financial accelerator effect could dominate the price effect (higher asset price increases firms' wealth and eases their credit constraint, which boosts demand of land and raises asset price further).

2.6 Asset market equilibrium

For the time being, we focus on the case where η is small so that the demand function is downward sloping. Small η, i.e. smaller financial accelerator coefficient, would be more plausible assumption, as firms normally do not mark-to-market their physical assets frequently to avoid their profits being exposed to market risks. The small η also helps us simplify the model, as we will see below.[5] We will see the case when η is large in Section 3.4.

Asset market equilibrium is determined as the solution to the following equations.

$$
\begin{aligned}
k_t^f &= \frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k_{t-1}^f\right\}}{q_t - (1-h)E[q_{t+1}]} \\
k_t^b &= \frac{A}{2} - \frac{1}{2}rq_t \\
K &= k_t^f + k_t^b
\end{aligned}
$$

---

[5] Note that this assumption is technically very similar to what Jeane and Korinek (2010) assume on the variable ###.

Since we have proved that the firms' constrained demand function is monotonically decreasing and the dealers' demand function (defined on k_t^f) is monotonically increasing, The existence of unique equilibrium is ensured. If the intersect of the functions are below the dotted line in Figure 2(a), i.e. $q_t < (1-h)E[q_{t+1}]$, then the equilibrium price is $q_t^* = (1-h)E[q_{t+1}]$. This equilibrium represents bubble as banks lend more than the collateral value by expecting the price increase. We will see below that this does not occur on any equilibrium pathes.

At t=2, q_3 is given exogenously as $E_2[q_3] = q_2$.

$$k_t^f = \frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k_{t-1}^f\right\}}{hq_t}$$

$$k_t^b = \frac{A}{2} - \frac{1}{2}rq_t$$

The same argument applies to ensure the existence of unique equilibrium. Trivially unique equilibrium exists at t=0 irrespective of η, as ω_0 is constant. The following proposition is the summary of the results.

Proposition 2 *For each period, the baseline model has unique equilibrium when η is small.*

We have the following lemma from the discussion above. The mathematical proof is provided at the appendix.

Lemma 3 k_t^f *and* q_t *are strictly increasing against* $\psi(T_{t-1})$.

Figure 2(a) summarises the proposition and the lemma (for given $E_t[q_{t+1}]$). The arrow shows what happens to the equilibrium if the economy experiences a negative macro shock. The negative income gain $(a_L - E_{t-1}[a_t])k_{t-1}^f$ shifts the demand curve leftward and the following decline of asset price generates the capital loss $\eta(q_t - E_{t-1}[q_t])k_{t-1}^f$. The capital loss reduces the net wealth further and lowers the demand curve further (this process will be repeated). This is the financial accelerator effect of the model.

2.7 Dynamics of the model

For the given parameter sets, we have seen that there exists unique equilibrium asset price. The model needs to specify the equilibrium profile $\left\{k_t^f, k_t^b, D_t, R_t, q_t\right\}$ for $t \in \{0, 1, 2\}$. At t=3, there
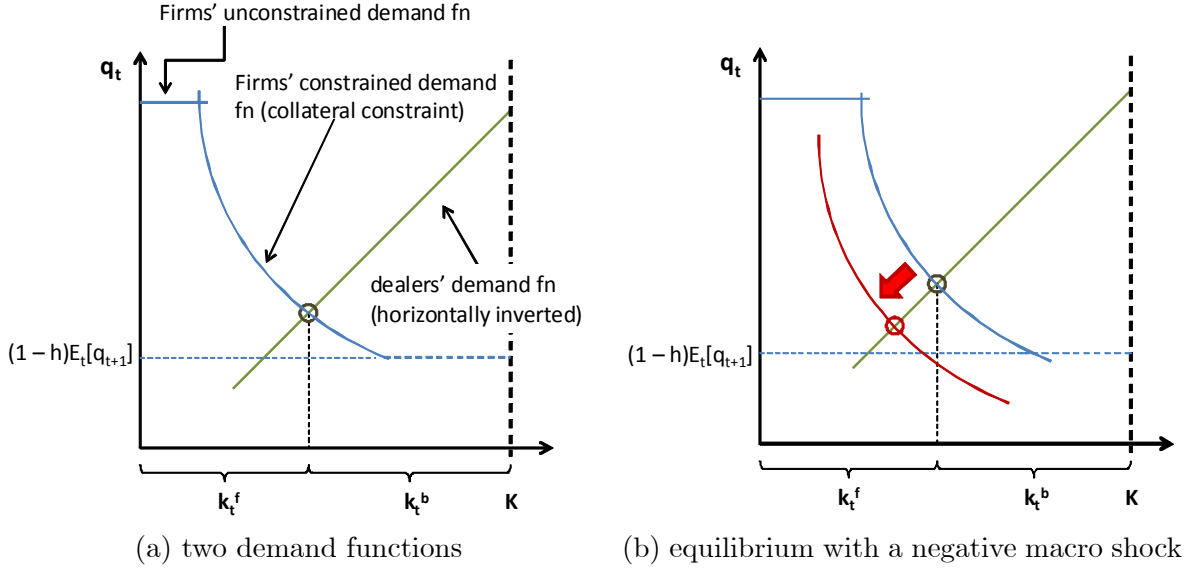
Figure 2: Equilibria with weaker financial accelerators

is no production opportunity and all players consume everything and die: therefore we do not need to specify anything other than q_3. In a closed economy, q_3^* has to be equal to zero as land loses its value as production capital. This is, however, not a convenient feature to consider asset price dynamism, which we focus on in the following section of forbearance. Therefore we assume an outsider of the economy visiting the economy at t=3 and purchases all land at the equilibrium price of the previous period; $q_3 = q_2^*$. This is clearly an ad-hoc assumption but it does not lose much generality as we will see below.

We solve the model by backward induction. Given that $q_3 = q_2^*$, we first solve the equilibrium at t=2. From the proposition y2, the uniqueness of q_2^* is ensured. Equilibrium q_2^* determines k_2^f and k_2^b simultaneously, which determines the equilibrium D_2. From the equation (6) and the firms' credit constraints, $E_2[\omega_3] = \omega_2 = (1-\gamma)T_2$: this specifies equilibrium R_2.

Given that the equilibrium action profile at t=2, we can proceed to the equilibrium at t=1. We do not need to consider complicated problems to maximise $E_1[\omega_3]$, $E_1[W_3]$ and $E_1[B_3]$ by the actions at t=1, as we have seen that firms' decisions are always constrained, banks' optimal decisions are equivalent to their myopic decisions maximising $E_1[W_2]$ and dealers are short-lived. From the firms' participation constraint, $E_1[\omega_2] = \omega_1$. If there is no anticipated uncertainty over a_2, $E_1[\omega_2] = \omega_1$ ensures that $E_1[q_2] = q_1$ because the dealers' demand function is identical over time and because of the lemma 3. Obviously, if $T_1 = \omega_1/(1-\gamma)$ and $E_1[q_2] = q_1$, then the firms' demand function at t=2 is equal to the one at t=1, which is consistent to the assumption

$E_1[q_2] = q_1$. The lemma 3 ensures that any other T_1 can satisfy the consistency. The exactly the same argument applies when we consider the equilibrium at t=0.

However, when we have anticipated uncertainty on a_t, $E_t\left[\pi\omega_t\left(a^H\right) + (1-\pi)\omega_t\left(a^L\right)\right]$ is not necessarily equal to ω_{t-1} when $T_{t-1} = \omega_{t-1}/(1-\gamma)$. This is because q_t is not a linear function of ω_t (and of a_t). This is obvious from the proof of the lemma 3. We thus need to adjust T_{t-1} to ensure the participation constraint binding, which resembles to certainty equivalence. In this section we determine the adjustment factor ζ_t. Readers may skip the following proposition since this is not an essential part of the model: assuming $\zeta_t = 0$ does not change the story.

Lemma 4 *In any possible equilibrium, k_2^{f*} and q_2^* are strictly increasing and concave with respect to T_1.*

See appendix for the proof. Since q_2^* is a concave mapping of T_1, q_2^* is also concave against the production shock $\Delta a_1 = a_1 - E[a]$. This means that ω_2 is a concave function of a_1, which ensures that $\pi\omega_t\left(a^H\right) + (1-\pi)\omega_t\left(a^L\right) < \omega_1$ when $T_1 = \omega_1/(1-\gamma)$. We thus need to increase T_1 to satisfy the participation constraint and the risk premium is denoted as ζ_1.

Since $E_1[\omega_2]$ is monotonically increasing wrt ζ_1, through itself and through a higher $E[q_2]$, there exists unique $\zeta_1^* > 0$ such that $E_1[\omega_2] = \omega_1$. With ζ_1^*, $E[q_2]$ has to be still smaller than q_1. This is because, if $E[q_2] \geq q_1$ with the ζ_1^*, there is no capital loss to be compensated, then we have to have $\zeta_1^* \leq 0$ in order to have $E_1[\omega_2] = \omega_1$. This is contradiction. The difference $\varphi_1 = E[q_2] - q_1$ is the function of the curvature of the demand curve, ω_1, and Δa_1: i.e. φ_1 is fixed when the auctioneer determines q_1. The same argument applies for $t = 0$ (see the appendix for the proof). We have proved the following proposition.

Proposition 5 *The asset price q_t follows a random walk process with a small negative drift $\varphi_t < 0$ under the optimal loan contract. $\varphi_2 = 0$ since there is no uncertainty in this period.*

Given that $E_t[q_{t+1}] = q_t + \varphi_t$, we can rewrite the land demand functions as follows, for any $t \in \{0, 1, 2\}$:

$$
\begin{aligned}
k_t^f &= \frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k_{t-1}^f\right\}}{hq_t - (1-h)\varphi_t} \\
k_t^b &= \frac{A}{2} - \frac{1}{2}rq_t \\
K &= k_t^f + k_t^b
\end{aligned}
$$

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At the period zero, no investment has been made and $k_{-1}^f = 0$.

Note that, on the equilibrium path, q_t^* and $y_t^* = a_t k_{t-1}^f$ are positively correlated, and Kiyotaki and Moore (1997) model.

3   Forbearance model

In the benchmark model, we assume that banks have to liquidate all seized collateral immediately, together with surviving firms and dealers. This has ensured that the total supply of land is always K, which is allocated by Walrasian auctioneer to firms and new-born dealers at the market price q_t. Banks could fail as a result, since the credit losses realised by the liquidation of collateral land reduces W_t lower than the threshold $\bar{W}$. Whether banks fail or not is not important to specify equilibrium shown above, since we do not assume any real economic cost of bank failure (since each bank's lending amount is not constrained, surviving banks can replace failed banks' loans immediately without any friction).

Now we introduce a new action as follows. The banks' managers, who incur a private utility loss X, do not have to liquidate seized collateral for a fraction $\theta \in [0, 1]$. I.e. the banks keep their non-performing borrowers knowing that they do not recover. This is called as forbearance, zombie lending, or evergreening loans. The bank managers would choose $\theta > 0$ hoping for the recovery of the land market in the next period so that they can liquidate the 'toxic' assets at a better price.

$\theta = 0$ is equivalent to the benchmark case where banks do not forbear liquidation at all. $\theta = 1$ means that the bank does not liquidate any of the failed borrower's asset. $\theta \in (0, 1)$ represents a partial liquidation. This would represent a couple of situations. First, banks do not liquidate bad borrowers, and the non-performing loan contracts are valued by the collateral value (e.g. if LTV ratio is 1.2, the bank has to realise 20% loss of the loan). Second, banks liquidate bad borrowers but do not liquidate the collateral asset, as observed in Spain recently. The un-liquidated collateral is valued at mark-to-market value. Note that mathematically those two are equivalent. In the following we take the former assumption.

To simplify the banks' strategic behaviour, we assume that banks can choose forbearance only at t=1, and they have to unwind at t=2. This is to ensure that banks unwind all forborne loans at a market price, otherwise we cannot observeproper price dynamics under forbearance.

Forbearance in this model allows insolvent firms to hold (a fraction of) their production capital θk_0^f. The market clearing condition at time t=1 is now $K = k_1^f + k_1^b + \theta \gamma k_0^f$, i.e.
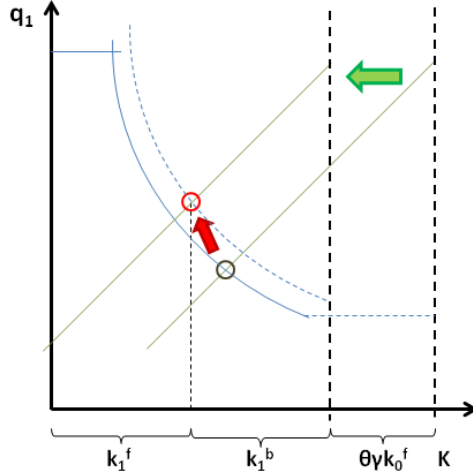
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Figure 3: The impact of forbearance

forbearance squeezes total land supply. In other words, this reduces the total supply of land temporarily at t=1 to $K - \theta\gamma k_0^f$, which shifts the dealers' demand curve to the left. We assume that θ is public information. For the time being we do not consider how the expectation of θ changes the equilibrium at t=0 for simplicity. I.e., we assume that the possibility of forbearance was not anticipated at all at t=0, which is not an unrealistic assumption.

Figure 3(a) describes how the equilibrium shifts by forbearance. The figure only show the initial impact alone assuming $E_1[q_2]$ is unaffected by θ, therefore the new crossing point is not necessarily the new equilibrium. Note that we have assumed small η. The details are discussed below.

The insolvent firms' recovery ratio is assumed to be zero for simplicity. Note that this is a neutral assumption, as we do not assume any recovery of these 'zombie' firms, nor the further deterioration of their credit quality. The expected change of the zombie firms' credit quality matters for the optimal choice of θ, but the absolute level of their credit quality does not.

3.1   Impact of θ

3.1.1   Price impacts

First we consider the impact of θ on q_1 and q_2.

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$$k_1^f = \frac{(1-\gamma)\left\{\psi(T_0) + \eta q_1(\theta) k_0^f\right\}}{q_1(\theta) - (1-h)E[q_2(\theta)]}$$

$$k_1^b = \frac{A}{2} - \frac{1}{2}r q_1(\theta)$$

$$K = k_1^f(\theta) + k_1^b(\theta) + \theta\gamma k_0^f$$

$$k_2^f = \frac{(1-\gamma)\left\{\psi(T_1(\theta)) + \eta q_2(\theta) k_1^f(\theta)\right\}}{h q_2(\theta)}$$

$$k_2^b = \frac{A}{2} - \frac{1}{2}r q_2(\theta)$$

$$K = k_2^f(\theta) + k_2^b(\theta)$$

By solving the set of equations for t=1, we have the following implicit function of $q_1(\theta)$, for given $E[q_2(\theta)]$:

$$\begin{aligned} & 2(1-\gamma)\psi(T_0) - 2\theta\gamma(1-h)E[q_2(\theta)]k_0^f \qquad (7) \\ = \ & r\{q_1(\theta)\}^2 - \left\{r(1-h)E[q_2(\theta)] + 2\theta\gamma k_0^f + 2\eta(1-\gamma)k_0^f\right\}q_1(\theta) \end{aligned}$$

Since the dealers' demand curve is always upward sloping, the leftward shift of the dealers' demand curve (by increasing θ) raises q_1 for sure, irrespective of the slope of the demand curve, for given and fixed $E[q_2(\theta)]$.

Since ω_1 is the positive function of q_1, ω_1 must be higher by forbearance and therefore it raises $E_1[\omega_2]$ to satisfy the firms' participation constraint $E_1[\omega_2] = \omega_1$. Higher $E_1[\omega_2]$ unambiguously raises $E_1[q_2]$ from the lemma 3, thus $E_1[q_2(\theta > 0)] > E_1[q_2(\theta = 0)]$ is ensured, which shifts the firms' demand curve upward at t=1 and t=2.

Next, we proceed to consider whether $E[q_2(\theta)]$ is bigger or smaller than $q_1(\theta)$, which is the crucial part of this section. Note first that, if θ kept constant at t=2, $E[q_2(\theta)] = q_1(\theta) + \varphi_1$ and $T_1 = \omega_1(\theta)/(1-\gamma) + \zeta_1$ as we have seen in the previous section. Note also that φ_1 and ζ_1 are independent from θ as θ does not change the curvature of the firms' demand function. Now consider $\theta = 0$ at t=2. $E[q_2(\theta)]$ has to be smaller than the case above as the dealers' demand function shifts rightward. This reduces $E_1[\omega_2]$ as the firms' capital loss is increased, which violates the firms' participation constraint followed by the increase of T_1. However, we also know that if T_1 is large enough to achieve $E[q_2(\theta)] = q_1(\theta) + \varphi_1$, then there is no capital loss and T_1 should go back to $T_1 = \omega_1(\theta)/(1-\gamma) + \zeta_1$. I.e. $T_1 = \omega_1(\theta)/(1-\gamma) + \zeta_1$ is too small as it violates

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the firms' participation constraint, and $T_1 > \omega_1(\theta)/(1-\gamma) + \zeta_1$ achieving $E[q_2(\theta)] = q_1(\theta) + \varphi_1$ is too large as it does not maximise banks' payoff (since the firms' participation constraint is not binding). Since $E[q_2(\theta)]$ is a continuous and monotonically increasing function of T_1 from the lemma 3, there exists unique T_1^* such that te firms' participation constraint is binding. And from the monotonicity of $E[q_2(\theta)]$ against T_1, we have the following proposition.

Proposition 6 $q_1(\theta)$ and $E[q_2(\theta)]$ are strictly increasing against θ. For any $\theta > 0$, $E[q_2(\theta)] < q_1(\theta)$.

Since the actual haircut $\tilde{h}$ is defined as $\tilde{h}(\theta) = 1 - (1-h)\frac{E[q_2(\theta)]}{q_1(\theta)}$, we have the following corollary.

Corollary 7 The banks' actual haircut $\tilde{h}$ is increasing against θ.

This corollary plays an important role when we consider how forbearance affects lending and output.

3.1.2 Output impacts

When we consider the impact of θ on y, we need to think two factors: one is the productive sector's k_t^f and the other is the decline of capital $(1-\theta)K$. In the following discussion we focus on the impact on k_t^f, since $\partial k_t^f/\partial\theta < 0$ is a sufficient condition of $\partial y/\partial\theta < 0$, and the size of k_t^f is crucial when we consider banks' profit function below. The impact on output y is shown by numerical exercise following.

We could see the sign of $\frac{\partial q_1}{\partial\theta}$ without endogenising $E[q_2;\theta]$ above, since upward shift of the demand curves by higher $E[q_2;\theta]$ unambiguously raise q_1. But it is unclear if the upward shift of the curves increases k_t^f, since it relies on the slope of the firms' demand curve: the sign of $\frac{\partial k_1^f}{\partial\theta}$ depends on how large those two demand curves shifts against the increase of θ.

If we endogenise $E[q_2;\theta]$ of the firms' demand function, then the firms' demand function no longer shifts against the change of θ, since θ affects firms' demand only through current and expected prices q_1 and $E_1[q_2]$. Now θ shifts the dealers' demand function only, and clearly the slope of the firms' demand function determines the sign of $\frac{\partial k_1^f}{\partial\theta}$. If the firms' demand curve is downward sloping, the leftward shift of the dealers' demand curve reduces the equilibrium output k_1^f, and vice versa. Therefore, $\frac{\partial k_1^f}{\partial\theta} < 0$ iif $\frac{\partial k_1^f(\theta)}{\partial q_1(\theta)} < 0$ and $\frac{\partial k_1^f}{\partial\theta} > 0$ iif $\frac{\partial k_1^f(\theta)}{\partial q_1(\theta)} > 0$. And we have the following proposition:

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Proposition 8 *There exists $\hat{\eta}$ such that for any $\eta < \hat{\eta}$ k_1^f is decreasing against θ. k_2^f is always increasing against θ.*

See the appendix for the proof. The reason why k_1^f can be increasing against θ nevertheless the firms' demand curve is downward sloping is that a higher θ raises ω_1 and shifts the demand curve toward upper right, which could outweigh the decline of k_1^f by the leftward shift of the dealers' demand curve. In other words, forbearance has two effects: the wealth effect increasing firms' wealth by raising capital gain (than it should be), and the haircut effect tightening firms' credit constraint by raising the actual haircut $\tilde{h}$. When η is high the wealth effect dominates the haircut effect (graphically, when the firms' demand curve is steep), k_1^f can be increased by forbearance. k_2^f is always increasing against θ as higher θ increases $E_1[\omega_2]$ for sure.

This proposition implies the possibility of welfare-boosting forbearance. If η is high, i.e. the financial accelerator effect of the economy is strong, the plunge of land price q_1 is too costly for the economy to bear, which justifies forbearance. Maintaining land price higher could boost investment (than it should have been) and output.

The following figures summarise the arguments above. The thin lines represent the equilibrium paths of the baseline model (dotted lines represent expectation). For example, Figure 4(a) shows that $E_0[q_1] \simeq q_0$ and $q_1|_{a_H}$ is higher than $E_0[q_1]$ and $q_1|_{a_L}$. As we have seen above, the land price follows $I(1)$ process. Since $q_3 = q_2$, it is independent from a_t. Forbearance (depicted by thick red lines) lifts q_1 higher. q_2 goes back to the neighbourhood of the baseline equilibrium path as banks unwind all forbearance, but $q_2(\theta > 0)$ is still higher than the baselines. This is because forbearance raises firms' net wealth ω_1, and anticipating the price fall at t=2 rational firms keep larger retained earning T_1 than the baseline (in other words, banks have to lower lending rate to satisfy the firms' participation constraint). Note that the thick dotted line, describing $E_1[q_2; \theta > 0]$, is downward sloping, as discussed above.

3.2 Banks' incentive

We have seen above the impact of forbearance on the economy for a given θ. Next we will endogenise the choice of θ, as an optimal choice of banks. Banks choose θ to maximise their future payoffs, as described below. In short, banks choose θ to protect their capital level W_t in this model. W_1 decreases if the borrowers of the bank go bankrupt under recession. And the regulatory threshold $\bar{W}$ is prone to be tightened when banks experience these negative shocks.[6] Both create an incentive for banks to make up their capital level W_t by forbearance, otherwise these banks are forced to close their business.

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[6] For example, Basel I was implemented in Japan in 1993, two years after the market crash. Basel III will be implemented following to the crisis in 2008.
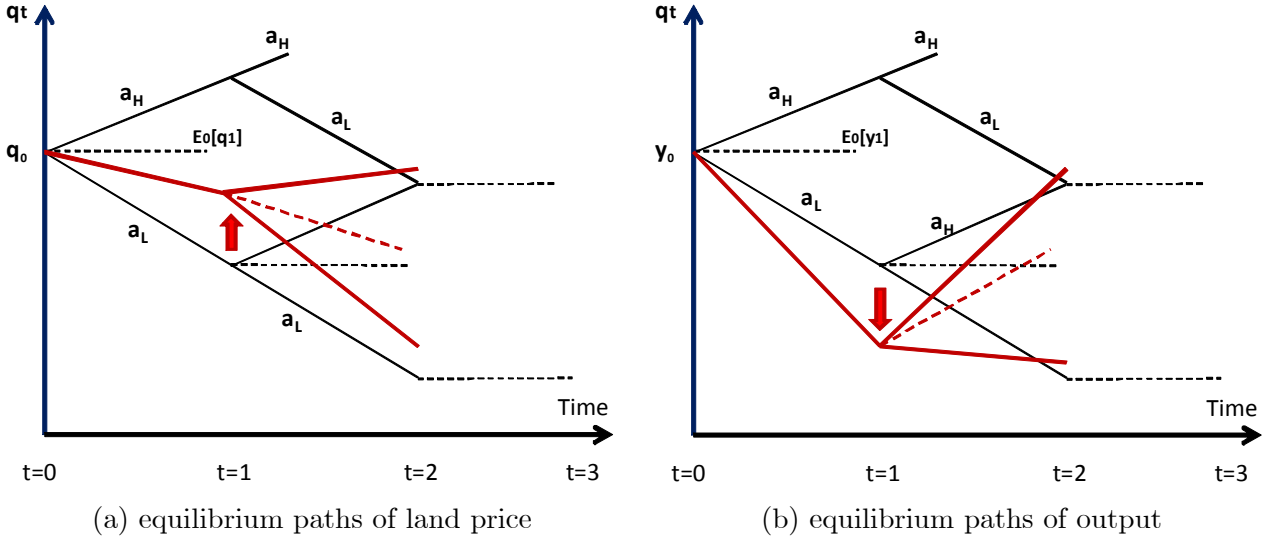
(a) equilibrium paths of land price (b) equilibrium paths of output

Figure 4: Equilibrium paths with and without forbearance

3.2.1 Defining stricken banks' capital

We have equilibrium q_t^{**} and k_t^{f**} for given θ, and the optimal θ is determined based on the equilibrium profile. Banks will choose their optimal θ, after a1 is chosen but before k_1^f, k_1^b and q_1 are determined, to maximise their final wealth W_3. First we define the capital of banks that experience default of the borrowers (stricken banks hereafter):

$$W_1\left(\theta\right) = W_0 - (1+r)D_0 + q_1\left(\theta\right)k_0^f \tag{8}$$

where the lending amount $D_t = q_t k_t^f - \omega_t$. The funding cost to support the unliquidated loans will be charged at the end of the period, i.e. at the beginning of t=2: we will come back to the assumption later. The last term is the value of forborne loans. As we have assumed above it is valued by the collateral value (i.e. they will record credit loss if the negative equity becomes larger than the haircut applied). The fraction $1 - \theta$ of the collateral value has been realised, and the rest of unliquidated collateral are evaluated at the same market price. Note that γ does not appear in the function since this is the payoff function of stricken banks only.

Lemma 9 $W_1\left(\theta\right)$ *is strictly increasing against* θ.

This is obvious from the definition. Higher θ raises q_1, which improves the banks' capital gain of their own investments k_0^b and the collateral value of failed firms $q_1\left(\theta\right)k_0^f$.

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The stricken banks choose θ to maximise their final expected payoff $E_1[W_3]$. As we discussed in section 2, this is not necessarily equivalent to maximise $E_1[W_2]$, since the model has two state variables ω_t and W_t which could affect future payoffs. Since W_t changes future payoffs only when $W_t < \bar{W}$, we can focus mainly on ω_t. If a bank chooses a higher θ, this would raise $E_1[\omega_2]$ and increase the profitable lending opportunity at t=2. Note that this is nearly identical to the issue in section 2, which banks can increase the profitable lending opportunity at t=2 by leaving some excess surplus on the hand of their borrowing firms. Even if a bank understand the possibility that raising θ can increase the profitable lending opportunity at t=2, the positive externality hinders the bank from raising θ to improve the lending opportunity. Therefore we can concentrate the following 'myopic' objective function without loss of generality.

At t=2 when the banks are forced to unwind the forbearance, banks see the cost of forbearance. The expected capital of the stricken banks W_2, conditional on $W_1 \geq \bar{W}$, is,

$$
\begin{aligned}
E_1[W_2] &= W_0 - (1+r)D_0 + (1-\theta)q_1(\theta)k_0^f \\
&\quad + (1-\gamma)\left(E[a_1]k_1^f(\theta) - T_1 - rD_1\right) + \gamma\left\{E_1[q_2(\theta)]k_1^f(\theta) - (1+r)D_1\right\} \\
&\quad + \theta E_1[q_2(\theta)]k_0^f - r\theta D_0 \\
&\quad + E_1[(1-I_2)X] \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (9)
\end{aligned}
$$

The first line is almost identical to W_1 except for the last term. The second line is the net expected profit of new loans at t=1 (forbearance does not hinder banks from new lending activity directly, since we do not assume capital adequacy ratio constraint). The third line is the part of profit or loss of forbearance: the first term is the liquidation revenue of the forborne loans and the second term is the funding cost. The fourth line is the expected cost of forced closure at t=2. Even if the bank can survive at t=1, it does not necessarily guarantee its survival in the following period. If the bank experiences another negative macro shock at t=2, and the borrowers go bankrupt again with probability γ, $W_2 < \bar{W}$ would be inevitable and the private cost should be taken into account (although this is not precisely a part of banks' capital).

Other than the net profit of new loans D_1 (the second line of the equation above), i.e. $(1-\theta)q_1(\theta)k_0^f + \theta E_1[q_2(\theta)]k_0^f - r\theta D_0$, can be increasing or decreasing against θ. The FOD of the first and the third line of the equation above is:

The following is the FOD of $E_1[W_2]$ w.r.t. θ. Note that the last term, $E_1[(1-I_2)X]$, is independent from θ. This is because we know that if the stricken banks fails to obtain repayments under recession $(a_2 = a_L)$ then the banks fail for sure, and otherwise they can survive (as what they obtain $E_1[a_2]k_2^f - T_2 - rD_2$ is large enough to survive from the assumption of sufficiently large $E[a]$).

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$$\frac{\partial E_1[W_2]}{\partial \theta} = (E_1[q_2(\theta)] - q_1(\theta))k_0^f + (1-\theta)\frac{\partial q_1}{\partial \theta}k_0^f + \theta\frac{\partial E_1[q_2]}{\partial \theta}k_0^f - rD_0$$

$$+ \{(1-\gamma)E[a_1] + \gamma E_1[q_2(\theta)]\}\frac{\partial k_1^f(\theta)}{\partial \theta} + \gamma\frac{\partial E_1[q_2(\theta)]}{\partial \theta}k_1^f(\theta) - (r+\gamma)\frac{\partial D_1}{\partial \theta}$$

The first line of the FOD is the marginal benefit of forbearance from forborne loans (i.e. the first and the third lines of the equation (9). The second line is the marginal benefit of forbearance from new loans D_1. It is not very easy to specify the sign of the FOD, as e.g. the marginal benefit of forborne loans w.r.t. θ can be positive or negative relying on the level of the bank's funding cost r; and the sign of $\partial D_1/\partial \theta$ is hard to specify as q_1 is increased but k_1^f decreases when $\eta < \check{\eta}$. However, since we have assumed the sufficiently large $E[a_1]$, the sign of $\partial E_1[W_2]/\partial \theta$ is in principle determined by the sign of $\partial k_1^f(\theta)/\partial \theta$. Now we have the following lemma:

Lemma 10 *If $E[a_t]$ is sufficiently large, $E_1[W_2]$ is strictly decreasing against θ when $k_1^f(\theta)$ is strictly decreasing against θ: i.e. when $\eta < \check{\eta}$.*

3.2.2 Optimal choice of forbearance

The stricken banks chooses θ to maximise $E_1[W_2]$. If $W_1 < \bar{W}$ the banks close immediately with the private cost X and W_2 is irrelevant. Since $X > E_1[(1-I_2)X]$ and the assumption that $E[a_t]$ is sufficiently large, the stricken banks do their best to avoid failure at t=1. In other words, since there is a chance to avoid failure at t=2 ($X > E_1[(1-I_2)X]$) and they can enjoy a profitable lending opportunity only when they survive at t=1, the stricken banks never choose to go bankrupt when they have a chance to survive by forbearance.

From lemma 9, W_1 is increasing against θ so that there exists unique $\hat{\theta}$ such that $W_1 = \bar{W}$. If $\hat{\theta} < 0$ then banks can survive without forbearance. If, instead, $\hat{\theta} > 1$ then banks cannot survive even if banks forbear. If $\hat{\theta} \in [0,1]$ then banks can survive at t=1 by choosing $\theta \geq \hat{\theta}$.

At the same time, from the lemma 10, $E_1[W_2]$ is decreasing against θ, i.e. forbearance is loss-making in the long run if $\eta < \check{\eta}$. If forbearance is loss-making, banks have an incentive to minimise it as long as it satisfies the survival condition $W_1 \geq \bar{W}$: i.e. $\theta^* = \hat{\theta}$ if $\hat{\theta} \in [0,1]$, and $\theta^* = 0$ otherwise.

However we have to consider here a potential strategic problem. In the previous argument we have implicitly assumed that all stricken banks choose the same θ so that q_1 is high enough

to achieve $W_1 = \bar{W}$, but there is no guarantee that $\theta^* = \hat{\theta}$ is supported as a Nash equilibrium. In the following, we will see what kind of θ can be supported as Nash equilibrium. Since all stricken banks are identical to each other, we focus only on symmetric Nash equilibria in the following.

First we consider if $\theta^* = \hat{\theta}$ is Nash equilibrium. Given that all the other stricken banks choose $\theta^* = \hat{\theta}$, the bank's optimal choice can be $\theta^* = 0$ if the bank is atomless: i.e. the bank's choice does not change q_1 at all, and therefore $\theta^* = \hat{\theta}$ cannot be the best response. However, we have assumed that the number of banks is 'finitely many'. If the number of banks N is any natural number, a bank's deviation from the action $\theta^* = \hat{\theta}$ lowers q_1 below the threshold maintaining $W_1 \geq \bar{W}$, triggering all stricken banks' bankruptcy including the deviator.[7] Therefore the deviation benefit and loss of a bank is, $\left(q_1^d - E\left[q_2\left(\hat{\theta} \right) \right] \right) k_0^f$ and $E_1\left[W_2\left(\hat{\theta} \right) \right] - \left\{ W_1\left(\hat{\theta} \right) + X \right\}$ respectively (where q_1^d is equilibrium land price when the bank chooses $\theta = 0$ and liquidates all forborne loans). The incentive compatibility constraint for this coalition of forbearance is:

$$\left(q_1^d - E\left[q_2\left(\hat{\theta} \right) \right] \right) k_0^f < E_1\left[W_2\left(\hat{\theta} \right) \right] - \left\{ W_1\left(\hat{\theta} \right) + X \right\}$$

where X is a negative number. There exists an sufficiently small (large negative) X for any sets of parameters (and $E[a]$ will be sufficiently high so that the higher X does not violate banks' participation constraint).

Second, if all the other stricken banks choose $\theta = 0$, a bank's optimal $\theta^* = 0$ when N is large enough since the bank alone cannot contain the plundge of q_1. The following proposition summarises the discussion.

Proposition 11 *If $k_1^f(\theta)$ is decreasing against θ, and if $\hat{\theta} \in [0,1]$, the stricken banks have two Nash equilibria, $\theta^{NE} = 0$ and $\theta^{NE} = \hat{\theta}$. If $\hat{\theta} \notin [0,1]$, $\theta^{NE} = 0$ is the unique equilibrium.*

The proposition above considers the case where $k_1^f(\theta)$ is decreasing against θ only: but we have seen that it is possible that $k_1^f(\theta)$ is increasing when η is high. In this case, from the lemma 10, forbearance becomes profitable in the long run. The unique Nash equilibrium is trivially $\theta = 1$, irrespective of the level of the stricken banks' capital.

Corollary 12 *If $k_1^f(\theta)$ is increasing against θ, the stricken banks' unique optimal choice is $\theta = 1$ irrespective of $\hat{\theta}$.*

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[7]Note that the argument here does not contradict with the discussion justifying the reason why banks do not leave excess surplus on the hand of firms (to maximise their long-run profit). Finite banks cannot maintain Nash equilibrium that all banks leave excess surplus on the hand of firms because the cost of deviation is proportional to the size of the banks and the deviation benefit easily outweights the cost. In this forbearance equilibrium, the deviation of a small bank triggers non-continuous change of the banks' utility function (by X).

This corollary also shows the feasibility of the 'output-boosting' forbearance. If the economy finds an opportunity of 'output-boosting' forbearance, the banks do it irrespective of their capital ratios. This contradicts with the observation by Peek and Rosengren (2005).

3.3 Anticipated forbearance before crisis

Throughout the discussions above, we have assumed that any players of the economy do not anticipate the possible forbearance. In this section we consider a situation when players are aware of the possibility of forbearance at t=0.

Anticipated forbearance changes the equilibrium at t=0 through a higher $E_0[q_1]$. Higher $E_0[q_1]$ raises expected wealth $E_0[\omega_1]$ of firms and reduces banks' actual haircut $\tilde{h}$. First, banks react by lowering firms' retained earning T_0, since the firms' participation constraint is not binding. From Lemma 3 we have unique T_0 achieving $E_0[q_1] = q_0 + \varphi_0$, but with such T_0 $E_0[\omega_1] < \omega_0$ since T_0 is lower than the previous cases nevertheless the additional capital gain was wiped off (this is analogous to the discussion of Proposition 6 using the intermediate value theorem). Therefore we have unique T_0 such that $E_0[\omega_1] = \omega_0$ and $E_0[q_1] > q_0 + \varphi_0$.[8]

The expected capital gain shifts firms' demand curve upward (through lower haircut $\tilde{h}$), which ensures the increase of q_0 and k_0^f. I.e. anticipated forbearance boosts pre-crisis economy by loosening credit constraint (lowering haircut). The boosted economy triggers a further vicious cycle: the increase of equilibrium q_0 and k_0^f raises banks' lending amount D_0, and this increases the banks' loss when they fail to be repaid in recession ($a_1 = a_L$), accompanied by the capital loss (as q_0 is higher). From the equation (8), higher q_0 and k_0^f raises the optimal $\theta^* = \hat{\theta}$, which further raises q_0 and k_0^f. The anticipation of forbearance itself amplifies the boom and the bust, in terms of output. Of course, the boom-bust cycle is less obvious in the land price by forbearance.

The following proposition summarises the above.

Proposition 13 *Anticipated forbearance will raise land price q_0 and k_0^f, as well as borrowers' debt outstanding D_0. This enlarges the loss of banks'that fail to receive repayments, raising $\hat{\theta}$.*

To be precise, the increased D_0 by the anticipation could raise $\hat{\theta}$ (see the equation 8) to be higher than 1. In this case $\theta^* = 0$ which makes the anticipation dynamically inconsistent. We do not consider the marginal cases in this paper.

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[8] Note that q_0 is independent from the choice of T_0 in this model.

3.4 Extension: in case demand function is upward sloping

So far, we have limited our attention to the case η is small, ensuring the downward sloping firms' demand curve. In this section we see the case to complete the analysis.

As the lemma 1 shows, small η ensures downward sloping demand curve but the opposite is not necessarily true: high η does not necessarily ensure upward sloping curve. But if the conditions are satisfied, i.e. $\eta > \hat{\eta}$ and $E_{t-1}[q_t] > (1-h)E_t[q_{t+1}]$, we have the following lemma.

Lemma 14 *If the firms' demand curve is upward sloping, the curve is convex ($\partial^2 q_t/\partial k_t^{f2}$).*

See the appendix for the proof (the proof of Lemma 1 has the result). The demand curve becomes upward sloping since the wealth effect (higher land price increases firms' net wealth) dominates the price effect (higher land price reduces the amount of land firms can purchase). Focusing on constrained equilibria, we could have multiple equilibria: the upper-right crossing is the stable equilibrium, and the lower-left crossing is the unstable equilibrium (Figure 5(a)). Note that the latter unstable equilibrium does not always exist (when the equilibrium q_t is below $(1-h)E_t[q_{t+1}]$). This is a reason why we limit our attention on the stable equilibrium below. The behaviours of equilibrium land price q_t and investment k_t^f against a negative macro shock are similar to the low η cases: the shock shifts the demand curve to the left, and both q_t and k_t^f decreases (see Figure 5(b)).

An interesting case of upward sloping demand curve is that if the demand curve shifts largely to the left and we lose the intersection with the dealers' demand curve. The equilibrium will jump discontinuously to $q_t = (1-h)E_t[q_{t+1}]$ or $q_t = 0$. The former exists only when we assume $E_t[q_{t+1}]$ as given and fixed, and it is not supported by the dynamic equilibrium, as $E_t[q_{t+1}] = q_t + \varphi_t$ on the equilibrium path. The latter is the case firms cannot borrow any from banks and stop investing at all (the assumption $A = 2K$ ensures that the dealers' demand curve goes through the origin, and firms are assumed not to invest without bank loans). This is a catastrophic equilibrium, although we do not study the details in this paper.

With the upward-sloping firms' demand curve, forbearance is unambiguously welfare-improving as it boosts both land price q_1 and the firms' investment k_1^f, and the optimal $\theta^* = 1$. In this sense, the cases of upward-sloping demand curve do not add much on the findings above. The mechanism of forbearance is basically identical to the case where the firms' demand curve is downward sloping but η is higher than $\check{\eta}$.

A potential additional benefit of forbearance in these cases is that forbearance could avoid catastrophic plunge of asset price. If a negative macro shock shifts the firms' demand curve to

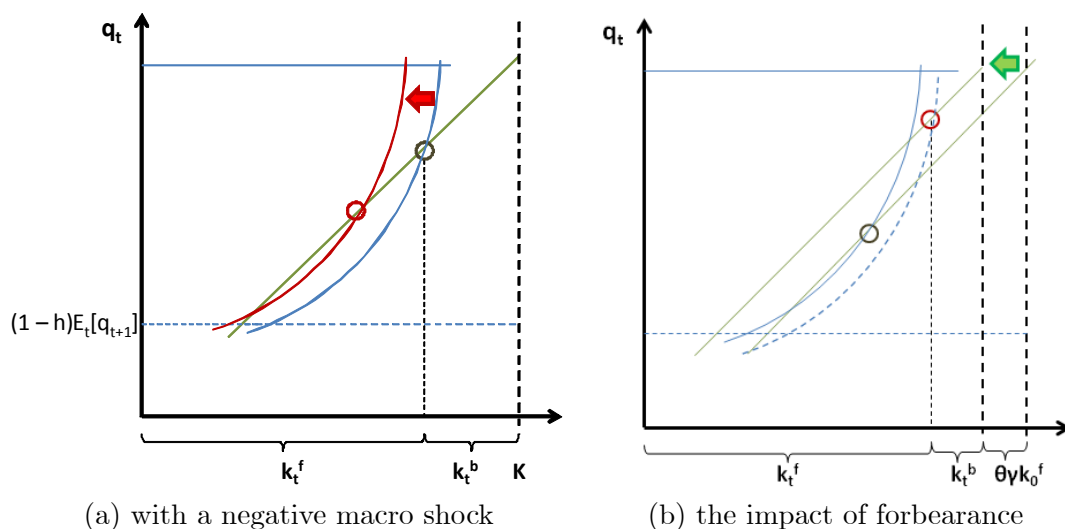(a) with a negative macro shock　　　　　(b) the impact of forbearance

Figure 5: Equilibria with stronger financial accelerators

the left so that it does not intersect with the dealers' demand curve, the equilibrium land price q_1 plunges to zero. Forberearance, shifting the dealers' demand curve to the left could recover the intersection of the two curves and the economy can avoid the severe market crash.

4　Forbearance and social welfare

Now we have seen the banks' strategic behaviour on forbearance and the resultant outcomes. We next see the social welfare of the identified equilibria to discuss possible policy options. In fact, this is a trivial question as we do not have any real cost of bank failure.

　Banks with $W_t < \bar{W}$ goes bankrupt when the equilibrium land price qt is chosen and W_t is realised. Faild banks are eliminated from the economy immediately, and the borrowers paired with these failed banks look for another lender replacing the loan contract. As we have discussed above, surviving banks can costlessly extend their loans to the borrowers since there is no constraint on the lending amount in this model (and each bank can lend to the infinite number of borrowers).[9]

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[9]Of course, in real life we may have real cost of banks' failure: e.g. borrowers may not be able to find a new lender in a timely fashion and they might need to cancel their investments. The operational cost of closing banks would not be negligible as well, and the depositors would incur the loss. Those factors would provide other reasons for social planner to choose regulatory forbearance. This is currently left for future study.

Since banks' failure is not costly for firms (and dealers) and the closure of banks is costly only for bank managers (private cost X), social planner's welfare function is thoroughly determined by the output y_t. To be precise, what we have seen above is the impact of forbearance to k_1^f and not output y_1, and higher k_1^f does not necessarily ensure the increase of y_1 since some land $\gamma\theta k_0^f$ is left unused. But here we do not distinguish these two to simplify the story as the difference is negligibly small when we assume very large $E[a_t]$.

Socially optimal θ is zero if forbearance is output-reducing (when η is small), and one if forbearance is output-boosting (when η is high). In other words, regulatory forbearance can be justified only when surviving firms' financial accelerator effect is very strong.

If the socially optimal θ is zero, public authorities would be required to consider policy options to discourage forbearance. The following section discusses a couple of policy options.

4.1 Policy options

We have seen that social planner finds it optimal to discourage or eliminate forbearance if the economy's financial accelerator is relatively weak, i.e. η is low. Here we discuss what kind of policy options the social planner has.

4.1.1 Capital injection

Since banks forbear liquidation to avoid the situation $W_1 < \bar{W}$, public capital injection increasing W_1 directly (or loosening capital requirement $\bar{W}$) would resolve the problem. The marginal effect of capital injection is, however, non-monotonic. In the equation (8) the capital injection can be described as increasing W_0. From Lemma 9 $\hat{\theta}$, the threshold to have $W_1 = \bar{W}$ is monotonically decreasing against the capital injection ΔW. Since the optimal $\theta^* = \hat{\theta}$ only when $\hat{\theta} \in [0,1]$ and $\theta^* = 0$ otherwise, $\hat{\theta}(\Delta W)$ is a discontinuous, non-monotonic function.

Insufficient capitalisation, or minimum capitalisation to keep stricken banks survive, therefore incentivises banks' forbearance. This is consistent to Philippon's and Schnabl's (2013) model and Giannetti's and Simonov's (2013) empirical findings.

4.1.2 Tight provisioning

Penalising forbearance, by tight provisioning or higher funding cost (to maintain non-performing loans), would have non-monotonic effects too. If public authorities require banks to increase loan loss provisions proportional to their zombie loans, we can rewrite the equation (8) as follows:

$$W_1(\theta) = W_0 - (1+r)D_0 + q_1(\theta)k_0^f - \beta\theta k_0^f$$

where β is the provisioning rate for a unit of zombie loan. Higher β obviously reduces the marginal benefit of forbearance $\partial W_1/\partial\theta$, and increases the threshold $\hat{\theta}$. Therefore the optimal θ^* is increasing against β as long as $\hat{\theta} \leq 1$, and then plunges to zero (see Figure **??** for the function of W_1).

We are able to discuss the impact of higher interest rate in a similar way. By raising r for banks' liability D_0, with a small modification to the model,[10] we have an analogous result to the tight provisioning.

Although both tight provisioning and capital injection have non-monotonic effects to forbearance, the nature of the effects is different. As the social planner increase capital injection to a bank, the optimal θ^* jumps to 1 from zero, and then gradually decreases until zero. Gradual tightening of credit loss provisions increases θ^* continuously, and then θ^* plunges from one to zero. Clearly, the social planner need to be cautious when it aims at discouraging forbearance by tightening provisioning or by higher funding cost, as it could trigger a sudden unwinding of forbearance in a massive scale.

4.1.3 Bad banks

If forbearance is welfare-boosting, on the other hand, regulatory forbearance is the optimal policy option as forbearance by banks is unstable. This is because forbearance is always one of multiple equilibria $\theta^* \in \left\{0, \hat{\theta}\right\}$, and it is not impossible that the equilibrium suddenly shifts from $\hat{\theta}$ to zero (e.g. by the hike of the funding cost). The government would find it optimal to stabilise the equilibrium by regulatory forbearance. The regulatory forbearance could take several different forms. For instance, ensuring low funding cost to banks can be a measure of forbearance, and establishing a special purpose vehicle purchasing non-performing loans and store those could be another option.

5 Conclusion

We have seen that why banks do forbearance and how it affects the economy using the reduced-form model of KM's (1997) Credit Cycles. Stricken banks cannot bear the cost of writing off bad

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[10]Currently the funding cost of forbearance is charged at the following period t=2 (i.e. implicitly assuming long-term funding), and does not appear in the equation above. In addition, interest rate also make dealers' demand curve flatter thus the impact would not be intuitive. But if we assume that higher interest rate affects banks alone and the funding cost is immediately charged at t=1, we have an analogous argument as the tight provisioning.

borrowers and forbear liquidating their non-performing loans to make up their balance sheets for the time being. This raises the price of collateral asset than it should be, and contains the negative equity of bad loans, allowing banks to make up their balance sheets.

Whether forbearance could increase the output of the economy or not relies on the structure of the economy. We have seen that if the financial accelerator effect of an economy is relatively weak, i.e. firms do not actively realise capital gains and losses of their asset holdings and maintain their investment levels stable, forbearance would lower the economy's output. This is because forbearance maintains land price higher than it should be, which reduces productive firms' "purchasing power" of land (price effect), and because the expectation of land price decline when banks unwind forbearance in the future tightens the credit constraint of firms through a higher haircut of collateral (haircut effect). Note that we have two channels of negative externalities here. First one is to productive firms: the haircut effect tightens the credit constraint of both productive and zombie firms. Second, banks lose their profitable lending opportunities to the productive firms — not only forbearing banks but healthy banks have the same problem.

The haircut effect plays an important role to endogenise leverage of the economy. In Kiyotaki and Moore (1997) and Krishnamurthy (2003), firms' net wealth is crucial in determining the amount of loans, as the leverage is given and fixed. This ensures the positive correlation between asset price and output. But in this model, anticipated decline of collateral value lowers leverage, as banks require a higher haircut of collateral to compensate the deterioration of collateral value. In other words, supported land price by forbearance creates pessimism over future land price, which reduces lending and output, resulting in the broken correlation.

But if the financial accelerator effect is strong, firms' investment becomes more sensitive to capital gains from land holding. Forbearance under recession contains the capital loss of the firms and support the firms' investment. If the financial accelerator effect is strong, this effect outweighs the other two effects above, and the output of the economy increases by forbearance. It is not easy to determine if an economy has a strong financial accelerator effect: it would be different across sectors (the accelerator of manufacturing industries would be low, while that of real estate sector would be high), different over time (e.g. the accelerator could be lower in recession) and different across countries. But if forbearance boosts output, banks should find it optimal to maximise forbearance as much as possible, which is not realistic. This would imply that forbearance is output reducing in most countries.

We have also seen banks' incentive mechanism of forbearance. Banks choose forbearance since it is too costly for the banks to realise loan losses, since the loss, lending amount minus the liquidation value of collateral, is large enough to lower their capitals below a regulatory

threshold and the banks go bankrupt. Forbearance is one of multiple equilibria, which are "no bank forbears" and "all stricken banks forbear". This could explain a reason why forbearance is not observed in the US.[11] Another possible reason of the domination of foreclosure in the US would be the less-monopolistic nature of lending contracts. We have assumed that banks can lend many firms but each firm can only borrow from a bank. This enables a bank to decide forbearance by itself. But if a firm is borrowing from many lenders, possibly through securitisation, each forbearance decision needs coordination of all lenders, which is extremely difficult. Some lenders would be well-capitalised and do not share the incentive of forbearance. In an extreme case, identifying all lenders would be difficult once a loan is securitised. Less-securitised lending contracts in Europe and Japan would make it easier for banks to forbear.

The welfare analysis of the paper is straightforward for now, as the failure of banks is costless for the economy. Therefore the optimal policy is discouraging forbearance (by tighter provisioning or capital injection) if forbearance reduces output, and encouraging forbearance (by lowering funding cost or a variation of bad banks) if forbearance increases output. The conclusion would change if we introduce a real economic cost of bank failure, e.g. by assuming that surviving banks cannot fully replace failed banks' loans. This is left for future studies for now.

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6   Appendix

6.1   Proof of Lemma 1

The FOD is,

$$
\begin{aligned}
\frac{\partial k_t^f}{\partial q_t} &= \frac{\eta(1-\gamma)k_{t-1}^f}{q_t - (1-h)E_t\left[q_{t+1}\right]} - \frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k_{t-1}^f\right\}}{\left\{q_t - (1-h)E_t\left[q_{t+1}\right]\right\}^2} \\
&= \frac{(1-\gamma)}{q_t - (1-h)E_t\left[q_{t+1}\right]}\left\{\eta k_{t-1}^f - \frac{\psi(T_{t-1}) + \eta q_t k_{t-1}^f}{q_t - (1-h)E_t\left[q_{t+1}\right]}\right\} \\
&= \frac{(1-\gamma)}{q_t - (1-h)E_t\left[q_{t+1}\right]}\left\{\frac{\eta\left\{-(1-h)E_t\left[q_{t+1}\right]\right\}k_{t-1}^f - \psi(T_{t-1})}{q_t - (1-h)E_t\left[q_{t+1}\right]}\right\}
\end{aligned}
$$

The (constrained) demand function is downward sloping if $-\eta(1-h)E\left[q_{t+1}\right]k_{t-1}^f - \psi(T_{t-1}) < 0$ and if $q_t - (1-h)E\left[q_{t+1}\right] \neq 0$.

$$-\eta(1-h)E\left[q_{t+1}\right]k_{t-1}^{f} - \psi(T_{t-1})$$

$$= -\eta(1-h)E\left[q_{t+1}\right]k_{t-1}^{f} - T_{t-1} - \left(a_t - E_{t-1}\left[a_t\right]\right)k_{t-1}^{f} + \eta E_{t-1}\left[q_t\right]k_{t-1}^{f}$$

$$= \eta\left\{E_{t-1}\left[q_t\right] - (1-h)E_t\left[q_{t+1}\right]\right\}k_{t-1}^{f} - \left(a_t - E_{t-1}\left[a_t\right]\right)k_{t-1}^{f} - T_{t-1} \qquad (10)$$

Since we have assumed that $T_{t-1} + \left(a_t - E_{t-1}\left[a_t\right]\right)k_{t-1}^{f} - \eta\left(q_t - E_{t-1}\left[q_t\right]\right)k_{t-1}^{f} > 0$ (to ensure that surviving firms with negative macro shock possess positive net wealth ω_t), if the first term is sufficiently small the demand function is downward sloping. If $E_{t-1}\left[q_t\right] < (1-h)E_t\left[q_{t+1}\right]$ the function is downward sloping irrespective of η. If $E_{t-1}\left[q_t\right] > (1-h)E_t\left[q_{t+1}\right]$, there exists unique $\hat{\eta}$ such that for any $\eta < \hat{\eta}$ the FOD is negative and vice versa. Note that $\hat{\eta}$ does not necessarily takes a feasible value, i.e. $\hat{\eta} \leq 1$.

The SoD is,

$$\frac{\partial k_t^{f2}}{\partial^2 q_t} = (-2)\frac{(1-\gamma)}{\left\{q_t - (1-h)E\left[q_{t+1}\right]\right\}^3}\left[\eta\left\{-(1-h)E\left[q_{t+1}\right]\right\}k_{t-1}^{f} - \psi(T_{t-1})\right]$$

The SOD is positive if the FOD is negative, and vice versa. I.e., the demand curve is concave if it is upward sloping, and is convex (to the origin) if it is downward sloping.

If $E_t\left[q_{t+1}\right] = q_t$, at t=2,

$$k_t^{f} = \frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k_{t-1}^{f}\right\}}{hq_t}$$

The FOD is,

$$\begin{aligned}
\frac{\partial k_t^{f}}{\partial q_t} &= \frac{(1-\gamma)\eta k_{t-1}^{f}}{hq_t} - h\frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k_{t-1}^{f}\right\}}{(hq_t)^2} \\
&= \frac{1-\gamma}{hq_t}\left\{\eta k_{t-1}^{f} - h\frac{\psi(T_{t-1}) + \eta q_t k_{t-1}^{f}}{hq_t}\right\} \\
&= \frac{1-\gamma}{(hq_t)^2}\cdot(-h)\cdot\psi(T_{t-1})
\end{aligned}$$

The FOD is negative if $h\psi(T_{t-1}) > 0$, i.e.

$$\begin{aligned}
-hT_{t-1} - h\left(a_t - E_{t-1}\left[a_t\right]\right)k_{t-1}^{f} + h\eta E_{t-1}\left[q_t\right]k_{t-1}^{f} &< 0 \\
\left\{\eta E_{t-1}\left[q_t\right] - \left(a_t - E_{t-1}\left[a_t\right]\right)\right\}k_{t-1}^{f} - T_{t-1} &< 0
\end{aligned}$$

For any parameter set, there exists a threshold $\hat{\eta}$ such that for any $\eta < \hat{\eta}$, the FOD is

negative. Note again that there is no guarantee that $\hat{\eta} < 1$. In other words, if $\psi(T_{t-1}) > 0$, then the demand function is downward sloping.

The SOD is,

$$\frac{\partial k_t^{f2}}{\partial^2 q_t} = \left(-2h^2\right)\frac{1-\gamma}{(hq_t)^3} \cdot (-h) \cdot \psi(T_{t-1})$$

The SOD is positive if $\psi(T_{t-1}) > 0$, i.e. if FOD is negative. This means that if the demand curve is upward sloping, it is convex, and if the curve is downward sloping it is convex to the origin. ■

6.2   Proof of Lemma 3

This is graphically obvious. We first prove $\partial k_2^f/\partial\psi\,(T_1)$ and then prove $\partial k_1^f/\partial\psi\,(T_0)$ by backward induction. From the Quadratic formula, the equilibrium k_2^{f*} should be:

$$k_2^{f*} = \frac{-\tilde{b} + \sqrt{\tilde{b}^2 + 4\tilde{a}\tilde{c}}}{2\tilde{a}}$$

The sign of the second term of the numerator should be plus in any cases, since if $\tilde{c} > 0$, $\frac{-\tilde{b}-\sqrt{\tilde{b}^2+4\tilde{a}\tilde{c}}}{2\tilde{a}}$ becomes negative, and if $\tilde{c} < 0$, $\tilde{b} < 0$ and $\tilde{b}^2 + 4\tilde{a}\tilde{c} > 0$, $\frac{-\tilde{b}-\sqrt{\tilde{b}^2+4\tilde{a}\tilde{c}}}{2\tilde{a}}$ is positive but unstable equilibrium we do not consider further. Clearly $\tilde{c}$ is strictly increasing against ψ, the lemma is proved for k_2^f and q_2.

At t=1,

$$\tilde{c} = (1-\gamma)\left\{\psi(T_{t-1}) + \eta\frac{E\left[q_{t+1}\right]}{1+r}k_{t-1}^f\right\}$$

Since we have just proved that $\partial E_1\left[k_2^f\right]/\partial\psi\,(T_1)$, $\tilde{c}$ is again strictly increasing against ψ. ■

6.3   Proof of Lemma 4

First we see the concavity of q_t against T_{t-1}. Since q_t is a linear function of k_t^f at the equilibrium (since the supply curve is linear) this is equivalent to see the FOD and SOD of k_t^f w.r.t. T_{t-1}. We see the equilibrium k_2^{f*} first as follows:

$$h\frac{2}{r}\left(k_2^{f*}\right)^2 - \eta(1-\gamma)\frac{2}{r}k_1^f k_2^{f*} = (1-\gamma)\psi(T_1)$$

Since we know that the sign of FOD is positive from Lemma 3, we will concentrate on the sign of SOD. The FOD of the implicit function is:

$$h\frac{4}{r}k_2^{f*}\frac{\partial k_2^{f*}}{\partial T_1} - \eta(1-\gamma)\frac{2}{r}k_1^f\frac{\partial k_2^{f*}}{\partial T_1} = (1-\gamma)\frac{\partial \psi(T_1)}{\partial T_1}$$

Note first that $\partial k_{t-1}^f/\partial T_{t-1} = 0$, since when a_t is realised k_{t-1}^f has been chosen and fixed. In addition, $\partial \psi(T_{t-1})/\partial T_{t-1} = 1$ by definition. Rearranging this, we have:

$$\left(h\frac{4}{r}k_2^{f*} - \eta(1-\gamma)\frac{2}{r}k_1^f\right)\frac{\partial k_2^{f*}}{\partial T_1} = 1-\gamma$$

The inside of the bracket of LHS has to be positive from Lemma 3. The SOD is:

$$h\frac{4}{r}\frac{\partial k_2^{f*}}{\partial T_{t-1}} + \left(h\frac{4}{r}k_2^{f*} - \eta(1-\gamma)\frac{2}{r}k_1^f\right)\frac{\partial^2 k_2^{f*}}{\partial T_{t-1}^2} = 0$$

If the FOD is positive the SOD has to be strictly negative at t=2.

Given this, we can consider the same concavity issue at t=1. Equilibrium k_1^{f*} is determined by the following equations:

$$k_1^f = \frac{(1-\gamma)\left\{\psi(T_0) + \eta q_1 k_0^f\right\}}{q_1 - (1-h)E\left[q_2\right]}$$
$$k_1^b = \frac{A}{2} - \frac{1}{2}rq_1$$
$$K = k_1^f + k_1^b$$

Substituting $E\left[q_2\right] = q_1 + \varphi_1$,

$$k_1^f = \frac{(1-\gamma)\left\{\psi(T_0) + \eta q_1 k_0^f\right\}}{hq_1 - (1-h)\varphi_1}$$
$$k_1^b = \frac{A}{2} - \frac{1}{2}rq_1$$
$$q_1 = \frac{A - 2\left(K - k_1^f\right)}{r}$$

Substituting this into the demand function, we have the equilibrium k_t^{f*}.

$$k_1^{f*} = \frac{(1-\gamma)\left\{\psi(T_0) + \eta\frac{A-2\left(K-k_1^f\right)}{r}k_0^f\right\}}{h\frac{A-2\left(K-k_1^f\right)}{r} - (1-h)\varphi_1}$$

$$\left\{h\frac{A-2\left(K-k_1^f\right)}{r} - (1-h)\varphi_1\right\}k_t^{f*} = (1-\gamma)\left\{\psi(T_0) + \eta\frac{A-2\left(K-k_1^f\right)}{r}k_0^f\right\}$$

$$\left\{\frac{A-2K}{1+r} - (1-h)\varphi_1 - \eta\frac{2}{r}(1-\gamma)k_0^f\right\}k_t^{f*} + h\frac{2}{r}\left(k_1^{f*}\right)^2$$
$$= (1-\gamma)\left\{\psi(T_{t-1}) + \eta\frac{A-2K}{r}k_{t-1}^f\right\}$$

Since $A = 2K$,

$$\left\{-(1-h)\varphi_1 - \eta\frac{2}{r}(1-\gamma)k_0^f\right\}k_t^{f*} + h\frac{2}{r}\left(k_1^{f*}\right)^2 = (1-\gamma)\psi(T_0) \qquad (11)$$

Taking derivative,

$$\left\{-(1-h)\varphi_1 - \eta\frac{2}{r}(1-\gamma)k_0^f\right\}\frac{\partial E_0\left[k_1^{f*}\right]}{\partial T_0} + h\frac{4}{r}E_0\left[k_1^{f*}\right]\frac{\partial E_0\left[k_1^{f*}\right]}{\partial T_0} = (1-\gamma)\frac{\partial E_1\left[\psi(T_0)\right]}{\partial T_0}$$

$$\left\{h\frac{4}{r}E_0\left[k_1^{f*}\right] - (1-h)\varphi_1 - \eta\frac{2}{r}(1-\gamma)k_0^f\right\}\frac{\partial E_0\left[k_1^{f*}\right]}{\partial T_0} = 1-\gamma$$

$$\left\{h\frac{4}{r}E_0\left[k_1^{f*}\right] - (1-h)\varphi_1 - \eta\frac{2}{r}(1-\gamma)k_0^f\right\}\frac{\partial^2 E_0\left[k_1^{f*}\right]}{\partial T_0^2} + h\frac{4}{r}\left(\frac{\partial E_0\left[k_1^{f*}\right]}{\partial T_0}\right)^2 = 0$$

$\frac{\partial E_0\left[k_1^{f*}\right]}{\partial T_0} > 0$ only when the bracket on LHS is strictly positive, and this is the sufficient condition for the second order derivative to be strictly negative. $E_0\left[k_1^{f*}\right]$ is therefore increasing and strictly concave wrt T_0. And from the firms' wealth transition function,

$$E_0[\omega_1] = (1-\gamma)\left\{\frac{\omega_0}{1-\gamma} + \eta E_0\left[q_1(a_0) - q_0\right]k_0^f\right\}$$
$$< \omega_0$$

a negative drift φ_0 is obtained. Since $E_0[\omega_2] = E_0[\omega_1]$, $E_0[\varphi_1] = \varphi_0$, although this $E_0[\varphi_1]$

never be realised since $\omega_2 \gtrless \omega_1$. ∎

6.4 Proof of Proposition 8

As explained above, we will see the condition to determine the sign of $\frac{\partial k_1^f}{\partial q_1}$, endogenising $E[q_2; \theta]$ of the following demand curve:

$$k_1^f = \frac{(1-\gamma)\left\{\psi(T_0) + \eta q_1(\theta) k_0^f\right\}}{q_1(\theta) - (1-h)E\left[q_2(\theta)\right]}$$

We will see the sign of $\frac{\partial k_1^f}{\partial q_1}$ for a given $\theta \geq 0$. Note that $E\left[q_2(\theta)\right]$ is not directly affected by θ: the price persistence of the model emerges only through ω_1, and ω_1 is influenced from θ only through q_1. Therefore we need to consider $\frac{\partial E[q_2(\theta)]}{\partial q_1(\theta)}$ only when we think about the behaviour of $E\left[q_2(\theta)\right]$, and we know $\frac{\partial E[q_2(\theta)]}{\partial q_1(\theta)} > 0$.

$$\frac{\partial k_1^f}{\partial q_1} = \frac{(1-\gamma)\eta k_0^f}{q_1(\theta) - (1-h)E\left[q_2(\theta)\right]} - \left\{1 - (1-h)\frac{\partial E\left[q_2(\theta)\right]}{\partial q_1(\theta)}\right\}\frac{(1-\gamma)\left\{\psi(T_0) + \eta q_1(\theta) k_0^f\right\}}{\{q_1(\theta) - (1-h)E\left[q_2(\theta)\right]\}^2}$$

Since $\frac{E[q_2(\theta)]}{q_1(\theta)} = \frac{E[q_2(\theta)]}{E[\omega_2(\theta)]} \cdot \frac{E[\omega_2(\theta)]}{q_1(\theta)}$, and $\frac{E[\omega_2(\theta)]}{q_1(\theta)} = \eta k_1^f(\theta)$, we can rewrite this as:

$$\frac{\partial k_1^f}{\partial q_1} = \frac{(1-\gamma)\eta k_0^f}{q_1(\theta) - (1-h)E\left[q_2(\theta)\right]} - \left\{1 - (1-h)\frac{E\left[q_2(\theta)\right]}{E\left[\omega_2(\theta)\right]} \cdot \eta k_1^f(\theta)\right\}\frac{(1-\gamma)\left\{\psi(T_0) + \eta q_1(\theta) k_0^f\right\}}{\{q_1(\theta) - (1-h)E\left[q_2(\theta)\right]\}^2}$$

multiply the RHS by $(1-\gamma)^{-1}\left\{q_1(\theta) - (1-h)E\left[q_2(\theta)\right]\right\}^2 > 0$, the RHS is:

$$\eta k_0^f\left\{q_1(\theta) - (1-h)E\left[q_2(\theta)\right]\right\} - \left\{\psi(T_0) + \eta q_1(\theta) k_0^f\right\} + (1-h)\frac{E\left[q_2(\theta)\right]}{E\left[\omega_2(\theta)\right]} \cdot \eta k_1^f(\theta)\left\{\psi(T_0) + \eta q_1(\theta) k_0^f\right\}$$

$$= (1-h)\frac{E\left[q_2(\theta)\right]}{E\left[\omega_2(\theta)\right]} \cdot \eta k_1^f(\theta)\left\{\psi(T_0) + \eta q_1(\theta) k_0^f\right\} - (1-h)E\left[q_2(\theta)\right]\eta k_0^f - \psi(T_0) \quad (12)$$

Since $\psi(T_0) > 0$ when $\eta = 0$, the RHS is strictly negative when $\eta = 0$. Since $\partial k_1^f/\partial q_1$ is a continuous function of η, there exists a threshold of η such that for any η smaller than the threshold $\partial k_1^f/\partial q_1 < 0$. As the increase of θ raises q_1, k_1^f is decreasing against θ in this region.

The impact of θ on k_2^f is relatively straightforward, since we know that $q_2(\theta > 0)$ is higher than $q_2(\theta = 0)$. This is thoroughly from a higher ω_1, and therefore higher q_2 is followed by higher k_2^f (irrespective of the level of η) (see Lemma 3). ∎